# Green currents for holomorphic automorphisms of compact Kähler manifolds

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February 8, 2008

**Abstract**

Let $f$ be a holomorphic automorphism of a compact Kähler manifold $(X,\omega)$ of dimension $k\geq 2$. We study the convex cones of positive closed $(p,p)$-currents $T_p$, which satisfy a functional relation

$$f^*T_p=\lambda T_p,\quad \lambda>1,$$

and some regularity condition (PB). Under appropriate assumptions on dynamical degrees we introduce closed finite dimensional cones, not reduced to zero, of such currents. In particular, when the topological entropy $\mathrm{h}(f)$ of $f$ is positive, then for some $m\geq 1$, there is a positive closed $(m,m)$-current $T_m$ which satisfies the relation

$$f^*T_m=\exp(\mathrm{h}(f))T_m.$$

Moreover, every quasi-p.s.h. function is integrable with respect to the trace measure of $T_m$. When the dynamical degrees of $f$ are all distinct, we construct an invariant measure $\mu$ as an intersection of closed currents. We show that this measure is mixing and gives no mass to pluripolar sets and to sets of small Hausdorff dimension.

**MSC:** 37F, 32H50, 32Q, 32U.
**Key-words:** Green current, equilibrium measure, mixing.

## 1 Introduction

Let $(X,\omega)$ be a compact Kähler manifold of dimension $k$. Let $f$ be a holomorphic automorphism of $X$. Our purpose is to introduce invariant positive closed currents for $f$ and to use them in order to construct dynamically interesting invariant measures.

More precisely we want to construct positive closed $(s,s)$-currents $T$ satisfying a functional equation

$$f^*T = \lambda T, \quad \lambda > 1,$$

and some additional regularity properties. Very likely, the currents $T$ will describe the distribution of invariant manifolds of codimension $s$ corresponding to the smallest Lyapounov exponents.

Let $d_p$ denote the dynamical degree of order $p$ of $f$. It describes the growth under iteration of the volume of $p$-dimensional manifolds. When $d_1 > 1$, it is natural to introduce first a positive closed $(1,1)$-current as

$$T_1 := \lim_{n\to\infty} \frac{(f^n)^*\omega}{d_1^n}.$$

Unfortunately, this limit does not always exist, as holomorphic automorphisms of tori show.

However, for Hénon maps in $\mathbb{C}^2$ or for algebraically stable meromorphic self-maps of $\mathbb{P}^k$ the limit exists and was studied extensively. See the introduction of [1, 35] for historical comments. The limit $T_1$ is called *the Green current*. For holomorphic endomorphisms of $\mathbb{P}^k$, the self-intersection $T_1^k$ is well defined and gives an invariant measure.

In the context of polynomial automorphisms $f$ of $\mathbb{C}^2$ having positive entropy, the second author has defined, using the intersection of the Green currents of $f$ and $f^{-1}$, an invariant probability measure which turned out to be dynamically interesting. It, as well as the Green current, was extensively studied by Bedford-Lyubich-Smillie [2, 1] and by Fornæss and the second author [22, 23].

Cantat [6] has adapted some of the constructions from the study of Hénon maps to the context of automorphisms of K3 surfaces (see also [28]). The striking fact in this case is the existence of automorphisms of positive entropy for some K3 surfaces [33, 6, 34].

Given a holomorphic automorphism or, more generally, a bimeromorphic self-map of a compact Kähler manifold of dimension $k \geq 3$, constructing interesting invariant measures is a delicate process. The case of some classes of polynomial automorphisms of $\mathbb{C}^k$ is considered in [35, 30]. The same difficulty arises for meromorphic self-maps whose topological degree is not the largest of the dynamical degrees, see [21, 28, 14]. In this case, the cohomology class $[T_1]^k$ is zero in the cohomology group $\mathcal{H}^{k,k}(X,\mathbb{R})$. Hence we cannot construct an invariant measure as the self-intersection of the Green current $T_1$.

In this paper, we focus on the question of constructing invariant currents and invariant measures for automorphisms of compact Kähler manifolds. We consider first automorphisms to avoid some technicalities. Some of the results can be extended to holomorphic maps, to non singular correspondences and to birational maps [19].

When $d_1 > 1$, we consider the convex cone $\Gamma_1$ of positive closed $(1,1)$-currents $T_1$ with bounded potentials such that $f^*T_1 = d_1 T_1$. We will show that this cone is closed, finite dimensional and not reduced to zero. Moreover, every current in $\Gamma_1$ has a Hölder continuous potential. An element of $\Gamma_1$ is called *a Green current of bidegree* $(1,1)$ (see [6, 28] for the case of surfaces).

To construct invariant currents of higher bidegree, we start with a positive closed current $T$ of bidegree $(s,s)$ satisfying

$$f^*T = \lambda_T T,\ \ \lambda_T > 0.$$

We introduce the cone $\Gamma(T)$ of positive closed $(s+1,s+1)$-currents $T\wedge S$ such that $T\wedge f^*S = \lambda_1(T) T\wedge S$ where $S$ is a closed (not necessarily positive) $(1,1)$-current with a continuous potential and

$$\lambda_1(T) := \lim_{n\to\infty} \left( \int T\wedge (f^n)^*\omega\wedge\omega^{k-s-1}\right)^{1/n}.$$

The number $\lambda_1(T)$ appears as a dynamical degree with respect to $T$. When $\lambda_1(T) > 1$, we show that $\Gamma(T)$ is a closed, finite dimensional, convex cone, not reduced to zero. This requires the introduction of cohomology groups with respect to $T$. We also prove that $\Gamma(T)$ is unchanged if we consider only currents $S$ with Hölder continuous potentials. In this approach, we use an inductive procedure, and at the step $s$ we replace the complex manifold $X$ by the $(s,s)$-current $T$.

When $d_1$ is strictly larger than the other dynamical degrees of $f$, using $f$ and $f^{-1}$, we can construct invariant positive currents as an intersection of closed $(1,1)$-currents. We first construct the current $T_1$ such that $f^*T_1 = d_1 T_1$ as above. We have $f_*T_1 = d_1^{-1}T_1$. We then construct $(1,1)$-currents $S_i$, $1\leq i\leq k-1$, with Hölder continuous potentials such that the currents $T_1\wedge S_1\wedge\ldots\wedge S_i$ are positive, closed and satisfy

$$f_*(T_1\wedge S_1\wedge\ldots\wedge S_i) = c_i T_1\wedge S_1\wedge\ldots\wedge S_i \text{ for some } c_i > 0.$$

In this case, we can show, at each step, that for the automorphism $f^{-1}$, the dynamical degree $\lambda_1(T_1\wedge S_1\wedge\ldots\wedge S_i)$, with respect to the current

$T_1 \wedge S_1 \wedge \ldots \wedge S_i$, is strictly larger than 1. The measure $\mu := T_k$ is invariant, mixing and gives no mass to pluripolar sets and to sets of small Hausdorff dimension.

In the general case, the method breaks down because the hypothesis on $\lambda_1(T)$ is not easy to check. Therefore, we introduce a second method for constructing directly an invariant current of bidegree $(s, s)$, under the assumption $d_s > d_{s-1}$. The current we construct is PB and satisfies

$$f^* T = d_s T. \tag{1}$$

Note that if a current $T$ is PB, it is weakly PB, i.e. q.p.s.h. functions are integrable with respect to the trace measure $T \wedge \omega^{k-s}$ (see Section 2.1 for details). The method uses a $\mathrm{dd^c}$-*resolution* as already used by the authors in various contexts [13, 16, 11]. We have to use complex analysis, i.e. estimates for the solutions of the $\mathrm{dd^c}$-equation (Proposition 2.1). This permits one to control the growth of $(f^n)_* \Phi$, where $\Phi$ is a test form. The delicate point in the construction is to obtain a PB current. See also [20] for another new method.

The method permits one to prove that $T$ is almost extremal, i.e. $T$ belongs to a finite dimensional extremal face of the cone of positive closed currents.

To construct an invariant measure $\mu$, we assume that the dynamical degrees of the automorphism $f$ are distinct. Then the Khovanskii-Tessier-Gromov concavity theorem implies the existence of an $m$ such that

$$1 < d_1 < \cdots < d_m > d_{m+1} > \cdots > d_k = 1.$$

The measure $\mu$ is then obtained using the first method for $f^{-1}$ but starting with $T_m$. Therefore, $\mu$ is a wedge product of $T_m$ with closed $(1, 1)$-currents with Hölder continuous potentials. We can choose $T_m$ so that $\mu$ is mixing and has positive Hausdorff dimension. Observe that according to Yomdin-Gromov [38, 26, 17, 18] the topological entropy $\mathrm{h}(f)$ of $f$ is equal to $\max \log d_i$. Hence, the PB current $T_m$ satisfies $f^* T_m = \exp(\mathrm{h}(f)) T_m$.

The classes of PC and PB currents, which we introduce, are of interest since they allow to define the product of currents of higher bidegree. We will come back to this subject in a furture work.

Many questions concerning Green currents and the measure $\mu$ are not studied here for instance: distribution of periodic points with respect to $\mu$, entropy of $\mu$, approximation of the Green currents by stable leaves... If $X$ is projective, the first author has proved that $T_m$ is weakly laminar [12].

The classification of manifolds of dimension $\geq 3$ with automorphisms of positive entropy is also an interesting problem. In dimension 2, many examples are given in [33, 6, 34]. Mazur's examples can be extended to dimension $\geq 3$.

**Mazur's examples.** Let $X$ be a smooth hypersurface of degree 2 of $\mathbb{P}^{1,k+1} := \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ ($k+1$ factors). Let $\pi_i$, $1 \leq i \leq k+1$, denote the $k+1$ projections of $X$ on the product of $k$ factors of $\mathbb{P}^{1,k+1}$. Assume that all $\pi_i$ are finite. Then, each fiber of $\pi_i$ contains exactly 2 points $z$, $z'$. We can define an involution $\tau_i$ by $z \mapsto z'$. The group generated by $\tau_i$ contains dynamically interesting automorphisms.

One can construct other examples by taking products of manifolds or the projectivization of their tangent bundle. The dynamics of these examples is however non trivial. It is used in [12] to get information on laminarity of currents. Examples on tori explain some of the difficulties that we have to overcome in the general case. Our results extend to non singular correspondences (see [32, p.337] and [7, 37, 11, 16] for definitions and examples).

## 2 Currents and groups of cohomology

A holomorphic automorphism $f$ of $X$ induces an invertible linear self-map on groups of cohomology. We will use this action of $f$ in order to construct invariant currents. We introduce classes of currents with some regularity properties.

We will write $u_n \simeq v_n$ for $\lim u_n/v_n = 1$. The notation $u_n \lesssim v_n$ means $\limsup |u_n/v_n| < +\infty$, the notation $u_n \sim v_n$ means $u_n \lesssim v_n$ and $v_n \lesssim u_n$, with the convention that $0 \simeq 0$, $0 \lesssim 0$ and $0 \sim 0$. For $(x, y)$ and $(x', y')$ in $\mathbb{R}^2$, we write $(x, y) \leq (x', y')$ if either $x < x'$, or if $x = x'$ and $y \leq y'$. The sign $\|\ \|$ denotes either the the mass of currents, the norm of a vector, or of a linear operator. The sign $[\ ]$ denotes a cohomology class of a closed current.

### 2.1 PC, PB and weakly PB currents

We refer to the survey by Demailly [8] for the basics on currents in complex analysis. Demailly's survey on Hodge theory [9] is also useful. We however recall a few facts.

When $(X, \omega)$ is a compact Kähler manifold of dimension $k$, a *current $T$ of bidegree* $(s, s)$ is a continuous linear form on $\mathcal{D}^{k-s,k-s}(X)$ – the space

of smooth forms of bidegree $(k-s, k-s)$. In a coordinate chart, $T$ can be expressed as a differential $(s,s)$-form with distribution coefficients. A $(k-s,k-s)$-form $\Phi$ is *weakly positive* if at every point $z \in X$

$$\Phi \wedge i\alpha_1 \wedge \overline{\alpha}_1 \wedge \ldots \wedge i\alpha_s \wedge \overline{\alpha}_s \geq 0$$

for every $(1,0)$-form $\alpha_j$ of $X$. The current $T$ is called *(strongly) positive* if $\langle T, \Phi\rangle \geq 0$ for every weakly positive test form $\Phi$. The space of currents is given the weak topology of currents. In particular, positive currents are currents of order zero. The *trace measure* $\sigma_T$ associated to a positive current $T$ is defined as $\sigma_T := \frac{1}{(k-s)!} T \wedge \omega^{k-s}$. The measure $\sigma_T$ is positive and the coefficients of $T$ in a chart are measures which are dominated by $c\sigma_T$, $c > 0$. We will denote by $\|T\|$ *the mass* $\int_X T \wedge \omega^{k-s}$ of $T$.

The calculus on differential forms extends to currents, except for the pullback by a holomorphic map, which is not a submersion. It is always delicate to define the wedge product of two currents. However when $u \in \mathrm{L}^1(\sigma_T)$, for example if $u$ is bounded, one can define $\mathrm{dd^c} u \wedge T := \mathrm{dd^c}(uT)$. The continuity properties of this operator depend on the properties of $u$ [8]. Recall that $\mathrm{d} = \partial + \overline{\partial}$, $\mathrm{d^c} = \frac{1}{2i\pi}(\partial - \overline{\partial})$ and that $\mathrm{dd^c} = \frac{i}{\pi}\partial\overline{\partial}$ is a real operator.

Let $T$ be a positive closed $(s,s)$-current, and $S$ be a closed $(1,1)$-current not necessarily positive. Since $X$ is Kähler, by Hodge theory, we can write $S = \alpha + \mathrm{dd^c} u$, where $\alpha$ is a smooth form cohomologous to $S$ and $u$ is a $(0,0)$-current. We say that $u$ is *a potential* of $S$. Observe that two potentials of $S$ differ by a smooth function. When $u$ is a $\nu$-Hölder continuous (resp. continuous or bounded) function, we say that $S$ has *a $\nu$-Hölder continuous* (resp. *a continuous or bounded*) *potential*. It is clear that this is independent of the choice of $\alpha$. For a current $S$ with a continuous potential, we can define $T \wedge S$ by $T \wedge S := T \wedge \alpha + \mathrm{dd^c}(uT)$. When $S$ is positive, we can choose $u$ upper semi-continuous. In this case, if $u$ is bounded, we can define $T \wedge S$ in the same way.

A real $(p,p)$-current $\Phi$ on $X$ is called *DSH* if $\Phi = \Phi_1 - \Phi_2$ and $\mathrm{dd^c}\Phi_i = \Omega_i^+ - \Omega_i^-$ where $\Phi_i$ are negative currents, and $\Omega_i^\pm$ are positive closed currents. We define the DSH-norm by

$$\|\Phi\|_{\mathrm{DSH}} := \inf\big\{\|\Phi_1\| + \|\Phi_2\| + \|\Omega_1^+\| + \|\Omega_2^+\|,\ \Phi_i,\ \Omega_i^\pm \text{ as above}\big\}.$$

Observe that $\|\Omega_i^+\| = \|\Omega_i^-\|$, and we can choose $\Phi_i$ and $\Omega_i^\pm$ such that $\|\Phi\|_{\mathrm{DSH}} = \|\Phi_1\| + \|\Phi_2\| + \|\Omega_1^+\| + \|\Omega_2\|$. Denote by $\mathrm{DSH}^p(X)$ the space of DSH $(p,p)$-currents. This is our space of test currents. A current $\Phi$ is in $\mathrm{DSH}^0(X)$ if and only if it is a Difference of q.p.S.H. functions.

Recall that an $\mathrm{L}^1$ function $\varphi : X \to \mathbb{R} \cup \{-\infty\}$ is *quasi-plurisubharmonic* (*q.p.s.h.* for short) if it is upper semi-continuous and $dd^c\varphi \geq -c\omega$, $c > 0$, in the sense of currents. A set $E \subset X$ is *pluripolar* if it is contained in the pole set $\{\varphi = -\infty\}$ of a q.p.s.h. function $\varphi$.

A topology on $\mathrm{DSH}^p(X)$ is defined in the following way: $\Phi^{(n)} \to \Phi$ in $\mathrm{DSH}^p(X)$ if $\Phi^{(n)} \to \Phi$ weakly and if $(\|\Phi^{(n)}\|_{\mathrm{DSH}})$ is bounded.

Let $T$ be a current of bidegree $(s, s)$ and of zero order. We say that $T$ is *PC* if it can be extended to a continuous linear form on $\mathrm{DSH}^{k-s}(X)$, and we write $\langle T, \Phi\rangle$ for the value of this linear form on $\Phi \in \mathrm{DSH}^{k-s}(X)$. In [18], we proved that every current in $\mathrm{DSH}^{k-s}(X)$ can be approximated by smooth forms. Then, the extension of $T$ is unique. Moreover, when $\Phi$ is continuous $\langle T, \Phi\rangle$ coincides with the usual integral [18, Prop. 4.6].

We say that $T$ is *PB* if there exists a constant $c_T > 0$ such that

$$|\langle T, \Phi\rangle| \leq c_T \|\Phi\|_{\mathrm{DSH}} \quad \text{for every DSH continuous } (k-s, k-s)\text{-form } \Phi.$$

The current $T$ is called *weakly PB* if

$$|\langle T, \varphi\omega^{k-s}\rangle| < +\infty \quad \text{for every q.p.s.h. function } \varphi \text{ on } X.$$

Observe that $T$ is weakly PB if and only if the measure $T \wedge \omega^{k-s}$ is weakly PB. If a positive current $T$ is (weakly) PB, then every positive current $T'$ such that $T' \leq T$, is also (weakly) PB. We can show that if $T$ is a positive weakly PB current, then $|\langle T, \varphi\omega^{k-s}\rangle| \leq c_T(1 + \|\varphi\|_{\mathrm{L}^1})$ for some constant $c_T > 0$ and for $\varphi$ q.p.s.h. such that $dd^c\varphi \geq -\omega$ (see [16] or Proposition 2.2).

In [13], we showed that a positive measure $\mu$ on a Riemann surface admits locally a Bounded Potential if and only if, locally p.s.h. functions are $\mu$-integrable. This justifies the term PB (see also Proposition 2.2).

The following result is useful in constructing PC and PB currents. It supplies the fact that when $\Omega^-$ is a smooth form cohomologous to a positive closed current $\Omega^+$, one cannot find, in general, a negative form $\Phi$ solving $dd^c\Phi = \Omega^+ - \Omega^-$. A counter-example can be founding in [4].

**Proposition 2.1** *There exists a constant $A > 0$ so that for every pair of positive closed $(k-s+1, k-s+1)$-currents $\Omega^\pm$ on $X$ with $[\Omega^+] = [\Omega^-]$, there are $\mathrm{L}^1$ negative $(k-s, k-s)$-forms $\Phi^\pm$ such that*

$$dd^c\Phi^+ - dd^c\Phi^- = \Omega^+ - \Omega^- \quad \textit{and} \quad \|\Phi^\pm\|_{\mathrm{DSH}} \leq A\|\Omega^+\|.$$

*Moreover, the DSH currents $\Phi^\pm$ depend continuously on $\Omega^\pm$. If $\Omega^\pm$ are continuous, then $\Phi^\pm$ are continuous.*

**Proof.** Analogous problems are considered in [18] and other aspects of the following computation are detailed there. By Hodge theory [25], we have

$$\mathcal{H}^{k,k}(X\times X,\mathbb{C})\simeq\sum_{\substack{p+p'=k\\ q+q'=k}}\mathcal{H}^{p,q}(X,\mathbb{C})\otimes_{\mathbb{C}}\mathcal{H}^{p',q'}(X,\mathbb{C}).$$

Hence, if $\Delta$ is the diagonal of $X\times X$, there exists a smooth real $(k,k)$-form $\alpha(x,y)$ on $X\times X$, cohomologous to $[\Delta]$ with $\mathrm{d}_x\alpha=\mathrm{d}_y\alpha=0$. Following Bost-Gillet-Soulé [24, 4], one can construct a $(k-1,k-1)$-form $K(x,y)$ on $X\times X$ such that $\mathrm{dd^c}K=[\Delta]-\alpha$. We recall the construction.

Let $\pi:\widehat{X\times X}\longrightarrow X\times X$ be the blow-up of $X\times X$ along $\Delta$. It follows from a theorem of Blanchard [3] that $\widehat{X\times X}$ is a Kähler manifold. Let $\widehat{\Delta}:=\pi^{-1}(\Delta)$ be the exceptional hypersuface. Choose a negative q.p.s.h. function $\widehat{\varphi}$ on $\widehat{X\times X}$ such that $\gamma:=-\mathrm{dd^c}\widehat{\varphi}+[\widehat{\Delta}]$ is a smooth form cohomologous to $[\widehat{\Delta}]$. We can choose [24, 1.3.6] a smooth closed real $(k-1,k-1)$-form $\eta$ on $\widehat{X\times X}$ such that $\pi^*\alpha$ is cohomologous to $[\widehat{\Delta}]\wedge\eta$, hence to $\gamma\wedge\eta$. It follows that there is a smooth real $(k-1,k-1)$-form $\beta$ such that $\mathrm{dd^c}\beta=-\gamma\wedge\eta+\pi^*\alpha$. Define

$$K(x,y):=\pi_*(\widehat{\varphi}\eta-\beta).$$

We have

$$\mathrm{dd^c}K(x,y)=\pi_*([\widehat{\Delta}]\wedge\eta-\pi^*\alpha)=\pi_*([\widehat{\Delta}]\wedge\eta)-\alpha.$$

The $(k,k)$-current $\pi_*([\widehat{\Delta}]\wedge\eta)$ is closed, of order zero and supported on $\Delta$. Hence, it is a multiple of $[\Delta]$. Moreover, $\pi_*([\widehat{\Delta}]\wedge\eta)$ is cohomologous to $[\alpha]=[\Delta]$. It follows that $\pi_*([\widehat{\Delta}]\wedge\eta)=[\Delta]$ and $\mathrm{dd^c}K=[\Delta]-\alpha$.

Since the forms $\eta$ and $\beta$ are smooth, we can write $\eta:=\eta^+-\eta^-$ and $\beta:=\beta^+-\beta^-$ with positive closed smooth forms $\eta^\pm$ and negative smooth forms $\beta^\pm$. Define $K^\pm:=\pi_*(\widehat{\varphi}\eta^\pm+\beta^\mp)$. These forms are negative and we have $K=K^+-K^-$. Moreover, there exist constants $c^\pm$ with $c^+-c^-=1$ and closed real $(k,k)$-forms $\Theta^\pm$ on $X\times X$ such that $\mathrm{dd^c}K^\pm=\Theta^\pm+c^\pm[\Delta]$. Let $|x-y|$ denote the distance between two points $x$ and $y$ of $X$ with respect to the Kähler metric on $X$. One can check that $K^\pm$, $\Theta^\pm$ are smooth on $X\times X\setminus\Delta$ and

$$K^\pm(x,y)=\mathrm{O}\big(|x-y|^{2-2k}\log|x-y|\big),\quad \Theta^\pm=\mathrm{O}\big(|x-y|^{2-2k}\big)$$

when $(x,y)\to\Delta$. This allows one to define

$$\Phi^+(x):=\int_{y\in X}K^+(x,y)\wedge\Omega^+(y)+\int_{y\in X}K^-(x,y)\wedge\Omega^-(y)$$

$$\Phi^-(x) := \int_{y\in X} K^+(x,y)\wedge\Omega^-(y) + \int_{y\in X} K^-(x,y)\wedge\Omega^+(y).$$

Since $dd^c K = [\Delta] - \alpha$, $d_y\alpha = 0$, and $\Omega^+ - \Omega^-$ is exact, we have

$$\begin{aligned} dd^c\Phi^+(x) - dd^c\Phi^-(x) &= \int_{y\in X} (dd^c)_x K(x,y)\wedge\big(\Omega^+(y)-\Omega^-(y)\big) \\ &= \int_{y\in X} dd^c K(x,y)\wedge\big(\Omega^+(y)-\Omega^-(y)\big) \\ &= \Omega^+(x)-\Omega^-(x) - \int_{y\in X}\alpha(x,y)\wedge\big(\Omega^+(y)-\Omega^-(y)\big) \\ &= \Omega^+(x)-\Omega^-(x). \end{aligned}$$

The description of the singularities of $K^\pm$ implies that $\|\Phi^\pm\|_{L^1}\lesssim\|\Omega^+\|$, that $\Phi^\pm$ depend continuously on $\Omega^\pm$ and that $\Phi^\pm$ are continuous when $\Omega^\pm$ are continuous. We can also write $\Theta^\pm$ as differences of positive closed forms, smooth on $X\times X\setminus\Delta$, with singularities of order $O(|x-y|^{2-2k})$. It follows that $\|\Phi^\pm\|_{DSH}\lesssim\|\Omega^+\|$. □

**Proposition 2.2** *Let $T$ be a positive $(s,s)$-current on $X$. If $T$ is PC, then $T$ is PB. If $T$ is PB, then it is weakly PB. A positive measure $\mu$ on $X$ is PB if and only if it is weakly PB.*

**Proof.** Let $\Phi_n$ be continuous forms with $\|\Phi_n\|_{DSH}=1$. If $c_n := |\langle T,\Phi_n\rangle|$ tend to $+\infty$, then $c_n^{-1}\Phi_n$ converge to 0 in $DSH^{k-s}(X)$, and $|\langle T, c_n^{-1}\Phi_n\rangle|$ converge to 1. Hence, $T$ is not PC.

Assume now that $T$ is PB. Let $\varphi$ be a strictly negative q.p.s.h. function on $X$ such that $dd^c\varphi\geq-\omega$. We have to show that $\langle T,\varphi\omega^{k-s}\rangle > -\infty$. Following a theorem of Demailly [10], there exists a decreasing sequence of smooth negative functions $(\varphi_n)$ with limit $\varphi$, which satisfy $dd^c\varphi_n\geq -c_X\omega$ where $c_X>0$ is a constant. We then have

$$|\langle T,\varphi\omega^{k-s}\rangle| = \lim|\langle T,\varphi_n\omega^{k-s}\rangle| \lesssim \limsup\|\varphi_n\omega^{k-s}\|_{DSH}\lesssim\|\varphi\|_{L^1}+1.$$

Hence $T$ is weakly PB.

Now, assume that $\mu$ is a weakly PB probability measure. We show that it is PB. If not, there would exist continuous functions $\varphi_n$ such that $\|\varphi_n\|_{DSH}=1$ and $\langle\mu,\varphi_n\rangle\geq n^3$. We can write $dd^c\varphi_n=\Omega_n^+-\Omega_n^-$ where $\Omega_n^\pm$ are positive closed $(1,1)$-currents such that $\|\Omega_n^\pm\|\leq 1$. Since the set of such currents is compact, there exist smooth forms $\Omega_n$, cohomologous to $\Omega_n^\pm$, such that $\Omega_n\leq c\omega$, where $c>0$ is a constant independent of $n$. There

exist q.p.s.h. functions $\varphi_n^\pm$ satisfying $\mathrm{dd^c}\varphi_n^\pm = \Omega_n^\pm - \Omega_n$ and $\max_X \varphi_n^\pm = 0$. The family of q.p.s.h. functions $\psi$ such that $\max_X \psi = 0$ and $\mathrm{dd^c}\psi \geq -c\omega$ is compact in $\mathrm{L}^1(X)$. Hence there is a constant $A > 0$ such that $\|\varphi_n^\pm\|_{\mathrm{DSH}} \leq A$. Define $c_n := \varphi_n - \varphi_n^+ + \varphi_n^-$. We have $\mathrm{dd^c} c_n = 0$. Hence $c_n$ is a constant and $|c_n| \lesssim \|\varphi_n\|_{\mathrm{L}^1} + \|\varphi_n^+\|_{\mathrm{L}^1} + \|\varphi_n^-\|_{\mathrm{L}^1} \leq 1 + 2A$. We then deduce, since $\varphi_n^+ \leq 0$, that

$$\langle \mu, \varphi_n^- \rangle = -\langle \mu, \varphi_n \rangle + c_n + \langle \mu, \varphi_n^+ \rangle \lesssim -n^3 + 1 + 2A.$$

It follows that $\langle \mu, \varphi \rangle = -\infty$ if $\varphi := \sum n^{-2} \varphi_n^-$. This is a contradiction because the series $\sum n^{-2} \varphi_n^-$ converges to a q.p.s.h. function. □

### 2.2 Some properties of linear maps

Recall that a *Jordan block* $J_{\lambda,m}$ is a square matrix $(a_{i,j})_{1 \leq i,j \leq m}$ such that $a_{ij} = \lambda$ if $i = j$, $a_{ij} = 1$ if $j = i+1$ and $a_{ij} = 0$ otherwise. If $\lambda \neq 0$, the entry of index $(1, m)$ of $J_{\lambda,m}^n$ is equal to $\binom{n}{m-1} \lambda^{n-m+1}$. This is the only entry of order $n^{m-1}|\lambda|^n$, the other ones have order at most $n^{m-2}|\lambda|^n$. We have

$$\|J_{\lambda,m}^n\| \sim \binom{n}{m-1} |\lambda|^{n-m+1} \sim n^{m-1}|\lambda|^n.$$

The eigenspace of $J_{\lambda,m}$ associated to the unique eigenvalue $\lambda$ is a complex line.

If $E$, $E'$ are real (resp. complex) vector spaces, denote by $\mathrm{End}(E, E')$ the space of $\mathbb{R}$-linear (resp. $\mathbb{C}$-linear) maps from $E$ onto $E'$. When $E$ and $E'$ are real vector spaces, we have

$$\mathrm{End}(E, E') \otimes_{\mathbb{R}} \mathbb{C} = \mathrm{End}(E \otimes_{\mathbb{R}} \mathbb{C}, E' \otimes_{\mathbb{R}} \mathbb{C}).$$

Hence, we can identify $\mathrm{End}(E, E')$ with a real vector subspace of $\mathrm{End}(E \otimes_{\mathbb{R}} \mathbb{C}, E' \otimes_{\mathbb{R}} \mathbb{C})$.

Consider a complex space $E$ and an invertible linear map $\Lambda \in \mathrm{End}(E, E)$. The space $E$ is the direct sum of the invariant complex subspaces $E = E_1 \oplus \cdots \oplus E_r$ with $\dim E_i = m_i$. The restriction of $\Lambda$ to $E_i$ is defined by a Jordan block $J_{\lambda_i, m_i}$. We can assume that $(|\lambda_1|, m_1) \geq \cdots \geq (|\lambda_r|, m_r)$. Let $F_i$ denote the eigenspace of $\Lambda_{|E_i}$. It is a complex line. Let $E_i'$ be the hyperplane generated by the first $(m_i - 1)$ vectors of the basis of $E_i$ associated to the Jordan form.

We say that $J_{\lambda_i, m_i}$ is *dominant* if $(|\lambda_i|, m_i) = (|\lambda_1|, m_1)$. In which case, we say that $\lambda_i$ is a *dominant eigenvalue* of $\Lambda$. Assume that $J_{\lambda_1, m_1}$, ..., $J_{\lambda_\nu, m_\nu}$ are the dominant Jordan blocks. It is clear that $\|\Lambda^n\| \sim n^{m_1 - 1}|\lambda_1|^n$

and that for any vector $v \notin E'_1 \oplus \cdots \oplus E'_\nu \oplus E_{\nu+1} \oplus \cdots \oplus E_r$, we have $\|\Lambda^n v\| \sim n^{m_1-1}|\lambda_1|^n$. The positive number $\lambda := |\lambda_1|$ is the *spectral radius* of $\Lambda$. The integer $m := m_1$ is called the *multiplicity of the spectral radius.*

We say that $F := F_1 \oplus \cdots \oplus F_\nu$ is the *dominant eigenspace* and that $F' := \oplus F_i$ with $1 \leq i \leq \nu$, $\lambda_i = \lambda$, is the *strictly dominant eigenspace* of $\Lambda$. These spaces are invariant under $\Lambda$. For any $1 \leq j \leq \nu$, there is a unique $\theta_j \in \mathbb{S} := \mathbb{R}/2\pi\mathbb{Z}$ such that $\lambda_j = \lambda \exp(i\theta_j)$. We say that $\theta := (\theta_1, \ldots, \theta_\nu) \in \mathbb{S}^\nu$ is the *dominant direction* of $\Lambda$; the dominant direction of $\Lambda^n$ is equal to $n\theta$. Denote by $\Theta$ the closed subgroup of $\mathbb{S}^\nu$ generated by $\theta$. It is a finite union of real tori. The orbit of each point $\theta' \in \Theta$ under the translation $\theta' \mapsto \theta' + \theta$ is dense in $\Theta$. If $\lambda_i = \lambda$ for every $1 \leq i \leq \nu$, we have $F = F'$, $\theta = 0$ and $\Theta = \{0\}$.

For every linear map $L = (L_1, \ldots, L_r)$ in

$$\mathrm{End}(E, E) = \mathrm{End}(E, E_1) \oplus \cdots \oplus \mathrm{End}(E, E_r)$$

let

$$\exp(-in\theta)L := \big(\exp(-in\theta_1)L_1, \ldots, \exp(-in\theta_\nu)L_\nu, L_{\nu+1}, \ldots, L_r\big).$$

We also define

$$\Lambda_n := \frac{\Lambda^n}{n^{m-1}\lambda^n} \quad \text{and} \quad \Lambda'_N := \frac{1}{N}\sum_{n=1}^{N} \Lambda_i.$$

Let $\pi : F \longrightarrow F'$ denote the canonical projection. The proof of the following proposition is left to the reader. One can deduce it from the proof of Proposition 2.4 if we take $K = E$ and $u = g = \mathrm{id}$.

**Proposition 2.3** *There exists a unique linear map $\Lambda_\infty : E \longrightarrow F$ such that*

$$\|\exp(-in\theta)\Lambda_n - \Lambda_\infty\| = \mathrm{O}(1/n) \quad \textit{and} \quad \|\Lambda'_N - \pi \circ \Lambda_\infty\| = \mathrm{O}(\log N/N).$$

*Moreover, $\pi \circ \Lambda_\infty$ is of rank $\dim F'$. If $\Lambda$ preserves a convex cone $\mathcal{K}$ which generates $E$ and satisfies $-\overline{\mathcal{K}} \cap \overline{\mathcal{K}} = \{0\}$, then $\lambda$ is a dominant eigenvalue of $\Lambda$ with an eigenvector in $\overline{\mathcal{K}}$.*

The last property of Proposition 2.3 is the Perron-Frobenius Theorem.

**Proposition 2.4** *Let $K$ be a metric space, and let $u : K \longrightarrow E$ be a $\nu$-Hölder continuous vector-valued function. Let $g : K \longrightarrow K$ be a Lipschitz*

*map such that $|g(x)-g(y)| \le M|x-y|$ where $M>1$ is a constant. Assume that the spectral radius $\lambda$ of $\Lambda$ is strictly larger than 1. Define*

$$v_n := \frac{1}{n^{m-1}\lambda^n}\sum_{j=1}^{n}\Lambda^j \circ u \circ g^{n-j} \quad \textit{and} \quad w_N := \frac{1}{N}\sum_{n=1}^{N} v_n.$$

*Then there exist vector-valued functions $v$ and $w$ on $K$ such that*

$$\|\exp(-in\theta)v_n - v\|_\infty = \mathrm{O}(1/n) \quad \textit{and} \quad \|w_N - w\|_\infty = \mathrm{O}(\log N/N).$$

*Moreover, $v$ and $w$ are $\nu'$-Hölder continuous for every $\nu'>0$ with $\nu' \le \nu$ and $\nu' < \log\lambda/\log M$.*

**Proof.** We can assume that the matrix $A$ of $\Lambda$ is the Jordan matrix $J_{\eta,m}$ with $\eta = \exp(i\theta)\lambda$ and $\theta \in \mathbb{R}/2\pi\mathbb{Z}$. Every entry of $A^n$ is of order at most $n^{m-2}\lambda^n$ except the one of index $(1,m)$ which is of order $n^{m-1}\lambda^n$. Since the functions $u\circ g^n$ are uniformly bounded, every entry of $A^n$, whose order is smaller or equal to $n^{m-2}\lambda^n$, does not contribute to $\lim v_n$ or $\lim w_N$. This follows from the estimate $\sum_{j=0}^n j^{m-2}\lambda^j \lesssim n^{m-2}\lambda^n$ for $\lambda>1$. Hence we can suppose that all the coordinate functions of $u$ are zero except the last one.

Let $u^+$ be the last coordinate function of $u$. Define $u_j^+ := u^+\circ g^j$ for $j \ge 0$ and $s_j := \binom{j}{m-1}\eta^{j-m+1}$ for $j \ge m-1$ with the convention that $\binom{0}{0} = 0$. The first coordinate functions of $v_n$ and $w_n$ are

$$v_n^+ := \frac{1}{n^{m-1}\lambda^n}\sum_{j=m-1}^{n} s_j u_{n-j}^+ \quad \text{and} \quad w_N^+ := \frac{1}{N}\sum_{n=1}^{N} v_n^+.$$

It is sufficient to study the sequences of functions $(v_n^+)$ and $(w_N^+)$.

We show first that $\|v_{n+1}^+ - \exp(i\theta)v_n^+\|_\infty \lesssim n^{-2}$. Observe that $\|u_j^+\|_\infty \le \|u^+\|_\infty$ and

$$S_{j,n} := \left|\frac{s_{j+1}}{(n+1)^{m-1}\lambda^{n+1}} - \frac{\eta s_j}{n^{m-1}\lambda^{n+1}}\right| \lesssim \frac{n-j+1}{n^2\lambda^{n-j}}.$$

Hence

$$\begin{aligned}
&|v_{n+1}^+ - \exp(i\theta)v_n^+| = \\
&= \left|\frac{1}{(n+1)^{m-1}\lambda^{n+1}}\sum_{j=m-2}^{n} s_{j+1}u_{n-j}^+ - \frac{1}{n^{m-1}\lambda^{n+1}}\sum_{j=m-1}^{n}\eta s_j u_{n-j}^+\right| \\
&\le \frac{|s_{m-1}|\|u_{n-m+2}^+\|_\infty}{(n+1)^{m-1}\lambda^{n+1}} + \sum_{j=m-1}^{n} S_{j,n}\|u_{n-j}^+\|_\infty \lesssim n^{-2}.
\end{aligned}$$

Thus, the sequence $\exp(-in\theta)v_n^+$ converges uniformly to a function $v^+$ and $\|\exp(-in\theta)v_n^+ - v^+\|_\infty = \mathrm{O}(1/n)$. It follows that $\|w_N^+ - w^+\|_\infty = \mathrm{O}(\log N/N)$ with $w^+ = 0$ if $\theta \neq 0$ and $w^+ = v^+$ if $\theta = 0$.

We only have to prove that $v^+$ is Hölder continuous. We will use the additive notation for the distance on $K$. Let $x, y \in K$ and $\delta := |x - y|$. First, consider the case where $\nu = \log\lambda/\log M$, so $\lambda = M^\nu$. Since $u^+$ is $\nu$-Hölder continuous, there exists $c > 0$, independent of $x$, $y$ such that

$$|u^+ \circ g^j(x) - u^+ \circ g^j(y)| \leq c|g^j(x) - g^j(y)|^\nu.$$

Hence

$$|u^+ \circ g^j(x) - u^+ \circ g^j(y)| \leq cM^{j\nu}\delta^\nu.$$

We also have $|u^+ \circ g^j(x) - u^+ \circ g^j(y)| \leq 2\|u^+\|_\infty$. Let $q$ be the integer part of $-\log\delta/\log M$. We have the following estimates:

$$\begin{aligned}
|v_n^+(x) - v_n^+(y)| &\leq \frac{1}{n^{m-1}\lambda^n} \sum_{j=m-1}^{n} s_j |u^+ \circ g^{n-j}(x) - u^+ \circ g^{n-j}(y)| \\
&\lesssim \frac{1}{n^{m-1}\lambda^n} \sum_{j=m-1}^{n} j^{m-1}\lambda^j |u^+ \circ g^{n-j}(x) - u^+ \circ g^{n-j}(y)| \\
&\lesssim \sum_{j=0}^{\infty} \lambda^{-j} |u^+ \circ g^j(x) - u^+ \circ g^j(y)| \\
&\lesssim \sum_{j=0}^{q} \lambda^{-j} M^{j\nu}\delta^\nu + \sum_{j=q+1}^{\infty} \lambda^{-j} \\
&\lesssim q\delta^\nu + \lambda^{-q} \lesssim -(\log\delta)\delta^\nu.
\end{aligned}$$

Consequently, $|v_n^+(x) - v_n^+(y)| \lesssim \delta^{\nu'}$. In the limit, we get $|v^+(x) - v^+(y)| \lesssim |x - y|^{\nu'}$. This is the required inequality.

The case where $\nu > \log\lambda/\log M$ is derived from the case $\nu = \log\lambda/\log M$. If $\nu < \log\lambda/\log M$, we have $M^\nu < \lambda$. In the same way, we obtain

$$|v_n^+(x) - v_n^+(y)| \lesssim \sum_{j=0}^{\infty} \lambda^{-j} M^{j\nu}\delta^\nu \lesssim \delta^\nu.$$

□

### 2.3 Action on cohomology groups

By Hodge theory [25, 9], we have the decomposition

$$\mathcal{H}^n(X,\mathbb{C}) = \sum_{p+q=n} \mathcal{H}^{p,q}(X,\mathbb{C})$$

where $\mathcal{H}^{p,q}(X,\mathbb{C})$ is the subspace of $\mathcal{H}^n(X,\mathbb{C})$ spanned by classes of closed $(p,q)$-forms. The group $\mathcal{H}^{p,q}(X,\mathbb{C})$ is isomorphic to the Dolbeault cohomology group of bidegree $(p,q)$. We also have $\mathcal{H}^{p,q}(X,\mathbb{C}) = \overline{\mathcal{H}^{q,p}(X,\mathbb{C})}$. Define

$$\mathcal{H}^{p,p}(X,\mathbb{R}) := \mathcal{H}^{p,p}(X,\mathbb{C}) \cap \mathcal{H}^{2p}(X,\mathbb{R}).$$

Then

$$\mathcal{H}^{p,p}(X,\mathbb{C}) = \mathcal{H}^{p,p}(X,\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}.$$

The *Kähler cone* $\mathcal{K}_X$ of $\mathcal{H}^{1,1}(X,\mathbb{R})$ is the cone of classes of Kähler forms on $X$. In what follows, $\mathcal{K}_X^*$ will denote the cone of classes of positive closed $(1,1)$-currents. These cones are convex; $\mathcal{K}_X$ is open; $\mathcal{K}_X^*$ is closed. We also have $\overline{\mathcal{K}}_X \subset \mathcal{K}_X^*$ and $(-\mathcal{K}_X^*) \cap \mathcal{K}_X^* = \{0\}$.

We will construct some invariant currents by induction starting from an invariant current. It is necessary to introduce cohomology groups relative to a positive closed current: the manifold $X$ is replaced by a current $T$. Let $T \neq 0$ be a positive closed current of bidegree $(s,s)$ with $0 \leq s \leq k-1$. For any $1 \leq p \leq k-s$, let $N^{p,p}(T,\mathbb{R})$ denote the space of classes $[\alpha] \in \mathcal{H}^{p,p}(X,\mathbb{R})$ which satisfy

$$[T] \wedge [\alpha] = 0 \ \text{ in } \ \mathcal{H}^{p+s,p+s}(X,\mathbb{R}).$$

Let $N^{1,1}_\nu(T,\mathbb{R})$ denote the space of classes $[S]$ where $S$ is a closed $(1,1)$-current (not necesarily positive) with a $\nu$-Hölder continuous potential such that $T \wedge S = 0$. Define

$$\mathcal{H}^{p,p}(T,\mathbb{R}) := \frac{\mathcal{H}^{p,p}(X,\mathbb{R})}{N^{p,p}(T,\mathbb{R})} \ \text{ and } \ \mathcal{H}^{1,1}_\nu(T,\mathbb{R}) := \frac{\mathcal{H}^{1,1}(X,\mathbb{R})}{N^{1,1}_\nu(T,\mathbb{R})}.$$

The map $\pi_T : \mathcal{H}^{p,p}(T,\mathbb{R}) \longrightarrow \mathcal{H}^{p+s,p+s}(X,\mathbb{R})$, given by $[\alpha] \mapsto [T] \wedge [\alpha]$, is injective. Hence $\|[T] \wedge .\|$ is a norm on $\mathcal{H}^{p,p}(T,\mathbb{R})$ if $\|.\|$ is a norm on $\mathcal{H}^{p+s,p+s}(X,\mathbb{R})$.

Let $f$ be a holomorphic automorphism of $X$ such that $f^*T = \lambda_T T$ where $\lambda_T > 0$ is a constant. We define the map $f^*$ on $\mathcal{H}^{p,p}(X,\mathbb{R})$ and $\mathcal{H}^{p,p}(T,\mathbb{R})$ by $f^*[\alpha] = [f^*(\alpha)]$ for every closed $(p,p)$-current $\alpha$ on $X$. Let

$$\lambda_{p,n}(T) := \lambda_T^{-n} \|[(f^n)^*(T \wedge \omega^p)]\| = \|[T \wedge (f^n)^* \omega^p]\|$$

and

$$\lambda_p(T) := \limsup_{n\to\infty} \lambda_{p,n}(T)^{1/n} = \limsup_{n\to\infty} \left( \int_X T \wedge (f^n)^* \omega^p \wedge \omega^{k-s-p} \right)^{1/n}.$$

Observe that $\lambda_p(T)$ depends neither on the Kähler form $\omega$ nor on the norm $\|.\|$ on $\mathcal{H}^{p+s,p+s}(X,\mathbb{R})$. The following proposition follows from the discussion on Jordan forms for linear maps and from Proposition 2.3 (see also [17, 18]).

**Proposition 2.5** *Let $X$, $f$ and $T$ be as above. Then, $\lambda_p(T)$ is the spectral radius of $f^*$ on $\mathcal{H}^{p,p}(T,\mathbb{R})$. If $l_p(T)$ is its multiplicity, then*

$$\lambda_{p,n}(T) \sim n^{l_p(T)-1} \lambda_p(T)^n.$$

*In particular, $\lambda_{p,n}(T)^{1/n}$ converge to $\lambda_p(T)$.*

**Proposition 2.6** *Let $X$, $T$, $f$ be as above. Then for every $p_1 \geq 1$ and $p_2 \geq 1$ such that $p_1 + p_2 \leq k - s$, we have $\lambda_{p_1+p_2}(T) \leq \lambda_{p_1}(T)\lambda_{p_2}(T)$. In particular, $\lambda_1(T)^p \geq \lambda_p(T)$ for $1 \leq p \leq k-s$ and $\lambda_1(T)^{k-s} \geq \lambda_T^{-1}$.*

**Proof.** Propositions 2.3 and 2.5 imply the existence of $[\alpha_1] \in \mathcal{H}^{p_1,p_1}(T,\mathbb{R})$, $[\alpha_2] \in \mathcal{H}^{p_2,p_2}(T,\mathbb{R})$ such that

$$\frac{[(f^n)^*\omega^{p_1}]}{n^{l_{p_1}(T)-1}\lambda_{p_1}(T)^n} \longrightarrow [\alpha_1] \quad \text{and} \quad \frac{[(f^n)^*\omega^{p_2}]}{n^{l_{p_2}(T)-1}\lambda_{p_1}(T)^n} \longrightarrow [\alpha_2].$$

Hence

$$\frac{[(f^n)^*\omega^{p_1+p_2}]}{n^{l_{p_1}(T)+l_{p_2}(T)-2}\lambda_{p_1}(T)^n\lambda_{p_2}(T)^n} \longrightarrow [\alpha_1] \wedge [\alpha_2]$$

in $\mathcal{H}^{p_1+p_2,p_1+p_2}(T,\mathbb{R})$. On the other hand, there exists a non-zero class $[\alpha] \in \mathcal{H}^{p_1+p_2,p_1+p_2}(T,\mathbb{R})$ such that

$$\frac{[(f^n)^*\omega^{p_1+p_2}]}{n^{l_{p_1+p_2}(T)-1}\lambda_{p_1+p_2}(T)^n} \longrightarrow [\alpha].$$

The property that $[\alpha] \neq 0$ implies that $\lambda_{p_1+p_2}(T) \leq \lambda_{p_1}(T)\lambda_{p_2}(T)$.

The inequality $\lambda_1(T)^p \geq \lambda_p(T)$ is clear. Since $f$ is an automorphism, the mass of the measure $(f^n)^*(T \wedge \omega^{k-s})$ is equal to the mass of $T \wedge \omega^{k-s}$. Thus, $\lambda_T \lambda_{k-s}(T) = 1$ and $\lambda_1(T)^{k-s} \geq \lambda_{k-s}(T) = \lambda_T^{-1}$. □

The space $\mathcal{H}^{1,1}_\nu(T,\mathbb{R})$ is invariant under $f^*$. Let $\rho_\nu(T)$ denote the spectral radius of $f^*$ on $\mathcal{H}^{1,1}_\nu(T,\mathbb{R})$ and $m_\nu(T)$ its multiplicity. Since $N^{1,1}_\nu(T,\mathbb{R})\subset N^{1,1}(T,\mathbb{R})$ we have

$$\big(\lambda_1(T),l_1(T)\big)\leq\big(\rho_\nu(T),m_\nu(T)\big).$$

We will prove later (see Lemma 3.3) that if $\lambda_1(T)>1$, and if $\nu$ is small enough, then the last inequality is in fact an equality.

When $T$ is the integration current on $X$, we define

$$d_{p,n}:=\|[(f^n)^*\omega^p]\| \quad\text{and}\quad d_p:=\lim_{n\to\infty}\sqrt[n]{d_{p,n}}.$$

We also have

$$d_p \quad=\quad \lim_{n\to\infty}\Big(\int_X (f^n)^*\omega^p\wedge\omega^{k-p}\Big)^{1/n}. \tag{2}$$

The numbers $d_p$ are called the *dynamical degrees* of $f$. The last one $d_t:=d_k$ is the *topological degree* of $f$ which is equal to 1 because $f$ is an automorphism. It is noticed in [29] that an inequality of Khovanskii-Teissier-Gromov shows that $p\mapsto\log d_p$ is concave, hence the sequence $(d_{p-1}/d_p)_{1\leq p\leq k}$ is increasing [31, 36, 27]. In particular, there exist $m$, $m'$, $1\leq m\leq m'\leq k-1$, such that

$$1\leq d_1<\cdots<d_m=\cdots=d_{m'}>\cdots>d_k=1.$$

From Proposition 2.5, we know that $d_p$ is the spectral radius of $f^*$ acting on $\mathcal{H}^{p,p}(X,\mathbb{R})$. Let $l_p$ denote its multiplicity. Since $f^*$ preserves the cone of classes of positive closed $(p,p)$-currents, by Proposition 2.3, $d_p$ is a dominant eigenvalue of $f^*$ on $\mathcal{H}^{p,p}(X,\mathbb{R})$. The relation

$$\int_X (f^n)^*\omega^p\wedge\omega^{k-p}=\int_X\omega^p\wedge(f^n)_*\omega^{k-p}$$

implies that the spectral radius of $f_*$ on $\mathcal{H}^{k-p,k-p}(X,\mathbb{R})$ is also equal to $d_p$ and its multiplicity is equal to $l_p$.

According to the Gromov-Yomdin theorem, the topological entropy $\mathrm{h}(f)$ of $f$ is equal to $\max_{1\leq p\leq k}\log d_p$ [26, 38, 17, 18]. In particular, if $\mathrm{h}(f)>0$, we have $\max d_p>1$. It follows from Proposition 2.6 that $d_1>1$. It is shown in [15] that the map which associates to a holomorphic endomorphism of $X$ its topological entropy has discrete image in $[0,+\infty[$. It is well known that if $f$ belongs to the component of the identity in the automorphism group of $X$, then $\mathrm{h}(f)=0$.

## 3 Relative Green currents

Let $f$ be a holomorphic automorphism of a compact Kähler manifold $(X, \omega)$ of dimension $k$. Let $T$ be a positive closed $(s,s)$-current, $0 \leq s \leq k-1$, on $X$, which satisfies a relation $f^*T = \lambda_T T$, $\lambda_T > 0$. When $s = 0$, $T$ is a multiple of the integration current on the manifold $X$ and $\lambda_T = 1$. Define

$$M_n := \|\mathrm{D}f^n\|_\infty \quad \text{and} \quad M := \lim_{n\to\infty} M_n^{1/n}$$

where $\mathrm{D}f^n$ is the differential of $f^n$. The constant $M$ is independent of the metric on $X$.

Assume that $\lambda_1(T) > 1$. Let $\Gamma(T)$ denote the cone of $(s+1, s+1)$-currents $T \wedge S$ where $S$ is a closed $(1,1)$-current (not necessarily positive) satisfying the following properties:

1. $S$ has a $\nu$-Hölder continuous potential for every $\nu$ such that $0 < \nu < \log \lambda_1(T)/\log M$;

2. $T \wedge S$ is a positive current;

3. $T \wedge f^*S = \lambda_1(T) T \wedge S$.

We will denote by $\mathbb{R}\Gamma(T)$ the real space generated by $\Gamma(T)$. We can now describe $\Gamma(T)$.

**Theorem 3.1** *Let $X$, $T$ and $f$ be as above. Assume that $\lambda_1(T) > 1$. Then, $\Gamma(T)$ is a closed finite dimensional cone with non zero elements. Let $R$ be a closed real $(1,1)$-current with a continuous potential. Then, the sequence of currents*

$$\frac{1}{N}\sum_{n=1}^{N} \frac{T \wedge (f^n)^* R}{n^{l_1(T)-1}\lambda_1(T)^n}$$

*converges to a current in $\mathbb{R}\Gamma(T)$, which depends only on the class $[R]$ in $\mathcal{H}^{1,1}(X, \mathbb{R})$. If $[R]$ belongs to $\mathcal{K}_X$, the limit current belongs to $\Gamma(T) \setminus \{0\}$.*

Fix $\nu$ such that $0 < \nu < \log \lambda_1(T)/\log M$. Replacing $f$ by $f^n$ with $n >> 0$, we can assume that $0 < \nu < \log \lambda_1(T)/\log M_1$. Let $m$ be the largest integer such that $[\omega]$, $[f^*\omega]$, $\ldots$, $[(f^{m-1})^*\omega]$ are linearly independent in $\mathcal{H}^{1,1}_\nu(T, \mathbb{R})$. Then there exist real numbers $a_0$, $\ldots$, $a_{m-1}$ such that we have in $\mathcal{H}^{1,1}_\nu(T, \mathbb{R})$

$$[(f^m)^*\omega] = a_{m-1}[(f^{m-1})^*\omega] + \cdots + a_0[\omega].$$

Let $E$ be the subspace of $\mathcal{H}^{1,1}_\nu(T,\mathbb{R})$ generated by $[\omega]$, ..., $(f^{m-1})^*[\omega]$. These $m$ classes form a basis $\mathcal{B}$ of $E$ and $E$ is invariant under $f^*$. Denote by $\Lambda$ the restriction of $f^*$ to $E$. The notations $\theta$, $\theta'$, $\Theta$, $\rho_\nu(T)$, $m_\nu(T)$ were introduced in Section 2.

The matrix of $\Lambda$ with respect to $\mathcal{B}$ is

$$A := \begin{pmatrix} 0 & 0 & \cdots & 0 & a_0 \\ 1 & 0 & \cdots & 0 & a_1 \\ 0 & 1 & \cdots & 0 & a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & a_{m-1} \end{pmatrix}.$$

Since $E$ contains a Kähler class, the spectral radius of $\Lambda$ and of $A$ are equal to $\rho_\nu(T) \geq \lambda_1(T) > 1$. Moreover, their multiplicities are equal to $m_\nu(T)$. We have

$$\|\Lambda^n\| = \|A^n\| \sim n^{m_\nu(T)-1}\rho_\nu(T)^n.$$

**Lemma 3.2** *There exist a closed $(1,1)$-current $S$ and a continuous family of closed $(1,1)$-currents $S_{\theta'}$ with $\nu$-Hölder continuous potentials, for $\theta' \in \Theta$, such that $T \wedge S \neq 0$, $T \wedge S_{\theta'} \neq 0$, $T \wedge f^*S = \rho_\nu(T) T \wedge S$ and $T \wedge f^*S_{\theta'} = \rho_\nu(T) T \wedge S_{\theta'+\theta}$. Moreover, the sequences of positive closed currents*

$$\widetilde{Z}_N := \frac{1}{N}\sum_{n=1}^{N} \frac{T \wedge (f^n)^*\omega}{n^{m_\nu(T)-1}\rho_\nu(T)^n} \quad \text{and} \quad Z_{n_i} := \frac{T \wedge (f^{n_i})^*\omega}{n_i^{m_\nu(T)-1}\rho_\nu(T)^{n_i}}$$

*converge to $T \wedge S$ and to $T \wedge S_{\theta'}$ when $N \to \infty$ and $n_i \to \infty$ with respectively $n_i\theta \to \theta'$. In particular, if $\Theta$ is reduced to one point, $Z_n$ converge to $T \wedge S$.*

**Proof.** From the definition of $\mathcal{H}^{1,1}_\nu(T,\mathbb{R})$, there exists a $(1,1)$-current $R$ with a $\nu$-Hölder continuous potential such that $T \wedge R = 0$ and

$$\Big[(f^m)^*\omega - \sum_{j=1}^{m} a_{m-j}(f^{m-j})^*\omega\Big] = [R] \ \text{ in } \ \mathcal{H}^{1,1}(X,\mathbb{R}).$$

Hence, there exists a $\nu$-Hölder continuous function $u$ such that

$$(f^m)^*\omega - \sum_{j=1}^{m} a_{m-j}(f^{m-j})^*\omega = R + dd^c u.$$

Then

$$T \wedge \left( (f^m)^*\omega - \sum_{j=1}^{m} a_{m-j}(f^{m-j})^*\omega \right) = T \wedge dd^c u.$$

Define

$$W_n := \begin{pmatrix} (f^n)^*\omega \\ \vdots \\ (f^{n+m-2})^*\omega \\ (f^{n+m-1})^*\omega \end{pmatrix} \quad \text{and} \quad U := \begin{pmatrix} 0 \\ \vdots \\ 0 \\ u \end{pmatrix}.$$

Then $W_{n+1} = f^*W_n$ and $T \wedge W_1 = T \wedge BW_0 + T \wedge dd^c U$ where $B$ is the transpose of $A$. By induction, we obtain

$$T \wedge W_n = T \wedge \left( B^n W_0 + dd^c \sum_{j=1}^{n} B^{j-1} U \circ f^{n-j} \right).$$

Define

$$\overline{W}_n := \frac{W_n}{n^{m_\nu(T)-1}\rho_\nu(T)^n}$$

and

$$V_n := \frac{1}{n^{m_\nu(T)-1}\rho_\nu(T)^n} \left( B^n W_0 + dd^c \sum_{j=1}^{n} B^{j-1} U \circ f^{n-j} \right).$$

Denote by $\overline{W}_n^+$ and $V_n^+$ the first components of $\overline{W}_n$ and of $V_n$. We have $[\overline{W}_n^+] = [V_n^+]$ in $\mathcal{H}_\nu^{1,1}(T, \mathbb{R})$. Proposition 2.3 implies that the sequence of classes $[\overline{W}_n^+]$ is bounded. Moreover, since $\omega$ is a Kähler form, any cluster point of of this sequence is a non-zero class.

Propositions 2.3 and 2.4 imply that when $(n_i\theta)$ converges to $\theta'$, the sequence $V_{n_i}^+$ converges to a current $S_{\theta'}$ with a $\nu$-Hölder continuous potential. Moreover, $S_{\theta'}$ depends on $\theta'$ but not on $(n_i)$. The current $T \wedge S_{\theta'}$ is positive and closed. Since $[S_{\theta'}] \neq 0$ in $\mathcal{H}_\nu^{1,1}(T, \mathbb{R})$, from the definition of $\mathcal{H}_\nu^{1,1}(T, \mathbb{R})$, we have $T \wedge S_{\theta'} \neq 0$. It is clear that $S_{\theta'}$ depends continuously on $\theta' \in \Theta$. Since $\lim(n_i + 1)\theta = \theta' + \theta$, we have

$$\begin{aligned} T \wedge f^*S_{\theta'} - \rho_\nu(T) T \wedge S_{\theta'+\theta} &= \lim \left( \frac{1}{\lambda_T} f^* Z_{n_i} - \rho_\nu(T) Z_{n_i+1} \right) \\ &= \rho_\nu(T) \lim \left( \left[ \frac{n_i+1}{n_i} \right]^{m_\nu(T)-1} Z_{n_i+1} - Z_{n_i+1} \right) \\ &= 0. \end{aligned}$$

Hence, $T \wedge f^* S_{\theta'} = \rho_\nu(T) T \wedge S_{\theta'+\theta}$.

Propositions 2.3 and 2.4 imply that the sequence of currents $\frac{1}{N}\sum_{n=1}^N V_n^+$ converges to a current $S$ which has a $\nu$-Hölder continuous potential. We also have $T \wedge S \neq 0$ because every limit value of $(T \wedge V_n^+)$ is a non-zero positive current. In the same way, we get $T \wedge f^* S = \rho_\nu(T) T \wedge S$. □

**Lemma 3.3** *Under the hypothesis of Theorem 3.1, we have*

$$\big(\rho_\nu(T), m_\nu(T)\big) = \big(\lambda_1(T), l_1(T)\big)$$

*for every $\nu$ such that $0 < \nu < \log \lambda_1(T)/\log M$.*

**Proof.** Recall that the map $\pi : \mathcal{H}^{1,1}(T, \mathbb{R}) \longrightarrow \mathcal{H}^{s+1,s+1}(X, \mathbb{R})$ defined by $\pi([\alpha]) := [T] \wedge [\alpha]$ is injective. Consequently, the spectral radius of $f^*$ on $\pi(\mathcal{H}^{1,1}(T, \mathbb{R}))$ is equal to $\lambda_T \lambda_1(T)$ and its multiplicity is equal to $l_1(T)$. We have seen that the sequence of classes

$$\frac{\big[(f^{n_i})^*(T \wedge \omega)\big]}{n_i^{m_\nu(T)-1} \lambda_T^{n_i} \rho_\nu(T)^{n_i}}$$

converges to $[T \wedge S_{\theta'}]$ in $\mathcal{H}^{s+1,s+1}(X, \mathbb{R}) \setminus \{0\}$. It follows from Proposition 2.5 that $\big(\rho_\nu(T), m_\nu(T)\big) = \big(\lambda_1(T), l_1(T)\big)$. □

**Lemma 3.4** *With the assumptions of Theorem 3.1, let $R$ be a closed $(1,1)$-current with a continuous potential such that $[R] = \lambda[\omega]$ in $\mathcal{H}^{1,1}(X, \mathbb{R})$ with $\lambda \in \mathbb{R}$. Then the sequence of currents*

$$\frac{1}{N} \sum_{n=1}^N \frac{T \wedge (f^n)^* R}{n^{l_1(T)-1} \lambda_1(T)^n}$$

*converges to $\lambda T \wedge S$. In particular, the limit is $0$ if $[R] = 0$ in $\mathcal{H}^{1,1}(X, \mathbb{R})$.*

**Proof.** Let $u$ be a continuous function such that $R = \lambda\omega + dd^c u$. We have

$$\frac{T \wedge (f^n)^* R}{n^{l_1(T)-1} \lambda_1(T)^n} = \lambda \frac{T \wedge (f^n)^* \omega}{n^{l_1(T)-1} \lambda_1(T)^n} + T \wedge dd^c \left( \frac{u \circ f^n}{n^{l_1(T)-1} \lambda_1(T)^n} \right).$$

Since the $u \circ f^n$ are uniformly bounded, the last relation implies that the sequence of currents

$$\frac{T \wedge (f^n)^* R}{n^{l_1(T)-1} \lambda_1(T)^n} - \lambda \frac{T \wedge (f^n)^* \omega}{n^{l_1(T)-1} \lambda_1(T)^n}$$

converges to 0. It suffices to apply Lemmas 3.2 and 3.3. The proof is valid for $R$ positive closed with a bounded potential. □

**End of the proof of Theorem 3.1.** Let $\omega_1, \ldots, \omega_j$ be Kähler forms such that the classes $[\omega_1], \ldots, [\omega_j]$ generate $\mathcal{H}^{1,1}(X, \mathbb{R})$. We can apply Lemma 3.4 to the forms $\omega_i$. This implies the convergence in Theorem 3.1.

We now show that $\dim \Gamma(T) \leq \dim \mathcal{H}^{1,1}(X, \mathbb{R})$. Otherwise, there exists a non-zero current $T \wedge S$ in $\mathbb{R}\Gamma(T)$ with $[S] = 0$. Let $u$ be a continuous function such that $S = \mathrm{dd^c} u$. We have

$$T \wedge S = \lim \frac{T \wedge (f^n)^* S}{\lambda_1(T)^n} = \lim T \wedge \mathrm{dd^c} \left( \frac{u \circ f^n}{\lambda_1(T)^n} \right) = 0.$$

This is impossible.

Let $T^{(1)}, \ldots, T^{(r)}$ be a maximal linearily independent set in $\Gamma(T)$. The cone $\Gamma(T)$ is equal to the intersection of the space generated by the $T^{(i)}$ and the cone of positive closed $(s+1, s+1)$-currents. Hence, it is closed. $\square$

When $T$ is the integration current on $X$, Propositions 2.3, 2.5, 2.6 and Theorem 3.1 imply the following result (see [6, 28] for the case of surfaces and [1, 23, 35] for polynomial automorphisms).

**Corollary 3.5** *Let $(X, \omega)$ be a compact Kähler manifold of dimension $k$. Let $f$ be a holomorphic automorphism of $X$, of positive topological entropy. Then $d_1 > 1$ and there exists a positive closed $(1,1)$-current $T_1$ satisfying $f^* T_1 = d_1 T_1$. Moreover, the potential of $T_1$ is Hölder continuous and the class $[T_1]$ belongs to $\overline{\mathcal{K}}_X$.*

**Remarks 3.6** The construction of invariant currents in Theorem 3.1 is still valid if we restrict to an invariant subspace $E$ of $\mathcal{H}^{1,1}(X, \mathbb{R})$. We have to assume that the spectral radius of $f^*$ on the projection $E'$ of $E$ in $\mathcal{H}^{1,1}(T, \mathbb{R})$ is strictly larger than 1. The construction gives invariant currents with Hölder continuous potentials which are not necessarily positive; there is a non-zero current if $f^*_{|E'}$ has a dominant real eigenvalue. In particular, every invariant $(1,1)$-current $T$ with continuous potential such that $f^* T = \lambda T$, $|\lambda| > 1$, has a Hölder continuous potential. Let $\mathcal{K}_X^b$ be the cone of non-zero classes of positive closed $(1,1)$-currents with bounded potentials. If $E \cap \mathcal{K}_X^b \neq \{0\}$, we obtain a positive current with a Hölder continuous potential.

# 4 Green currents

Let $f$ be as in Section 3. Recall that $d_s$ is the spectral radius of $f^*$ on $\mathcal{H}^{s,s}(X, \mathbb{R})$. Let $l_s$ denote its multiplicity. We consider $\Gamma_s$ the cone of PB

positive closed $(s,s)$-currents $T$ such that $f^*T = d_s T$. Let $\mathbb{R}\Gamma_s$ denote the real space generated by $\Gamma_s$. For every positive closed $(s,s)$-current $T$, define

$$\mathcal{C}(T) := \big\{S \text{ positive closed } (s,s)\text{-current}, \ S \leq cT \text{ for some } c > 0\big\}$$

and $\mathbb{R}\mathcal{C}(T)$ the real space generated by $\mathcal{C}(T)$. Let $[\ ]$ denote the map which associates to a closed $(s,s)$-current its cohomology class in $\mathcal{H}^{s,s}(X,\mathbb{C})$.

**Theorem 4.1** *Assume that $d_s > d_{s-1}$. Let $S$ be a PB closed real $(s,s)$-current. Then, the sequence of currents*

$$S_N := \frac{1}{N}\sum_{n=1}^{N} \frac{(f^n)^*S}{n^{l_s-1}d_s^n}$$

*converges to a current in $\mathbb{R}\Gamma_s$ which depends only on the class $[S]$ in $\mathcal{H}^{s,s}(X,\mathbb{R})$. Moreover, $\Gamma_s \neq \{0\}$ and the restriction of $[\ ]$ to $\mathbb{R}\Gamma_s$ is injective. Every current in $\mathbb{R}\Gamma_s$ is PC. If $T$ belongs to $\Gamma_s$, then the restriction of $[\ ]$ to $\mathbb{R}\mathcal{C}(T)$ is injective. In particular, the cones $\Gamma_s$ and $\mathcal{C}(T)$ are finite dimensional and closed. If $[T]$ is extremal in the cone of classes of positive closed $(s,s)$-currents then $T$ is extremal in the cone of positive closed $(s,s)$-currents.*

**Proof.** Let $\Phi$ be a DSH $(k-s,k-s)$-current such that $dd^c\Phi = \Omega^+ - \Omega^-$ where $\Omega^\pm$ are positive closed $(k-s+1,k-s+1)$-currents. Then $[\Omega^+] = [\Omega^-]$ and $\|\Omega^+\| = \|\Omega^-\|$. Assume that $\|\Omega^+\| = \|\Omega^-\| \leq 1$. Proposition 2.1 implies the existence of a $(k-s,k-s)$-form $\Phi_0 = \Phi_0^+ - \Phi_0^-$ such that $\Phi_0^\pm \leq 0$, $dd^c\Phi_0 = dd^c\Phi$ and $\|\Phi_0^\pm\|_{\mathrm{DSH}} \leq A$. The current $\Psi_0 := \Phi - \Phi_0$ is $dd^c$-closed. Define $\Omega_n^\pm := (f^n)_*\Omega^\pm$. Recall that the spectral radius of $f_*$ on $\mathcal{H}^{k-s+1,k-s+1}(X,\mathbb{C})$ is equal to $d_{s-1}$ and that its multiplicity is equal to $l_{s-1}$ (see Section 2). Fix $\epsilon$, $0<\epsilon<d_s-d_{s-1}$. We have $\|\Omega_n^\pm\| \lesssim (d_s-\epsilon)^n$.

Proposition 2.1 implies the existence of $(k-s,k-s)$-forms $\Phi_n = \Phi_n^+ - \Phi_n^-$ such that $dd^c\Phi_n = \Omega_n^+ - \Omega_n^-$, $\Phi_n^\pm \leq 0$ and $\|\Phi_n^\pm\|_{\mathrm{DSH}} \lesssim (d_s-\epsilon)^n$. If $S$ is PB and $\Phi$ is smooth or if $S$ is smooth and $\Phi$ is DSH as above, we have $|\langle S, \Phi_n^\pm\rangle| \lesssim (d_s-\epsilon)^n$ for every $n\geq 0$. We define by induction the $dd^c$-closed form $\Psi_n$ as $\Psi_n := f_*\Phi_{n-1} - \Phi_n$. They satisfy $\|\Psi_n\|_{\mathrm{L}^1} \lesssim (d_s-\epsilon)^n$ for $n\geq 1$. On the other hand, we have

$$(f^n)_*\Phi = (f^n)_*\Psi_0 + (f^n)_*\Phi_0 = (f^n)_*\Psi_0 + (f^{n-1})_*\Psi_1 + (f^{n-1})_*\Phi_1.$$

So by induction, we get

$$(f^n)_*\Phi = (f^n)_*\Psi_0 + \cdots + f_*\Psi_{n-1} + \Psi_n + \Phi_n.$$

Since $X$ is Kähler, every closed form which is d-exact is $dd^c$-exact [9, p.41]. Hence the $dd^c$-closed form $\Psi_n$ defines a linear form on $\mathcal{H}^{s,s}(X,\mathbb{R})$ by $[\alpha] \mapsto \int \Psi_n \wedge \alpha$ for every real closed $(s,s)$-form $\alpha$. The Poincaré duality allows to associate to $\Psi_n$ a unique class $c_n$ in $\mathcal{H}^{k-s,k-s}(X,\mathbb{R})$. For $n \geq 1$ we have

$$\|c_n\| \lesssim \|\Psi_n\|_{L^1} \lesssim (d_s - \epsilon)^n.$$

Define

$$b_n := (f^n)_* c_0 + (f^{n-1})_* c_1 + \cdots + c_n \quad \text{and} \quad B_N := \frac{1}{N} \sum_{n=1}^{N} \frac{b_n}{n^{l_s-1} d_s^n}.$$

As in the proof of the Proposition 2.4, we can check that the sequence $(B_N)$ converges to a class $B \in \mathcal{H}^{k-s,k-s}(X,\mathbb{C})$ such that $\|B\| \leq c\|\Phi\|_{\mathrm{DSH}}$ where $c > 0$ is a constant.

Now, assume that $S$ is smooth and $\Phi$ is DSH. Since $S$ is closed, then

$$\langle (f^n)^* S, \Phi \rangle = \langle S, (f^n)_* \Phi \rangle = \int [S] \wedge b_n + \langle S, \Phi_n \rangle \tag{3}$$

and

$$\langle S_N, \Phi \rangle = \int [S] \wedge B_N + \frac{1}{N} \sum_{n=1}^{N} \frac{\langle S, \Phi_n \rangle}{n^{l_s-1} d_s^n}.$$

The second term in the right hand side of the last equality tends to zero because $\|\Phi_n\|_{\mathrm{DSH}} \lesssim (d_s - \epsilon)^n$. Hence

$$\lim \langle S_N, \Phi \rangle = \int [S] \wedge B \leq c\|\Phi\|_{\mathrm{DSH}}.$$

It follows that $(S_N)$ converges to a PB current $S_\infty$ which depends only on the class $[S]$. It is clear that $f^* S_\infty = d_s S_\infty$. Hence $S_\infty$ belongs to $\mathbb{R}\Gamma_s$ (we can write $S$ and $S_\infty$ as differences of positive closed currents). Observe that if $S$ is strictly positive, by definition of $d_s$ and $l_s$, we have $[S_\infty] \neq 0$. Hence $S_\infty$ is a non-zero positive current and $\Gamma_s \neq \{0\}$.

Now assume that $S$ is PB (not necessarily smooth) and $\Phi$ is smooth. Then, $\Phi_n$ is continuous. If $S'$ is a smooth real $(s,s)$-form cohomologous to $S$, we have

$$\begin{aligned}
\langle S_N - S'_N, \Phi \rangle &= \int [S - S'] \wedge B_N + \frac{1}{N} \sum_{n=1}^{N} \frac{\langle S - S', \Phi_n \rangle}{n^{l_s-1} d_s^n} \\
&= \frac{1}{N} \sum_{n=1}^{N} \frac{\langle S - S', \Phi_n \rangle}{n^{l_s-1} d_s^n}
\end{aligned}$$

The last term tends to zero because $S - S'$ is PB and $\|\Phi_n\|_{\mathrm{DSH}} \lesssim (d_s - \epsilon)^n$. It follows that $(S_N)$ converges to a PB current in $\mathbb{R}\Gamma_s$.

Let $R \in \mathbb{R}\Gamma_s$ be a current such that $[R] = 0$. Then, using identity (3) we get

$$|\langle R, \Phi\rangle| = d_s^{-n}|\langle (f^n)^* R, \Phi\rangle| = d_s^{-n}|\langle R, \Phi_n\rangle| \lesssim d_s^{-n}(d_s - \epsilon)^n.$$

Therefore, $\langle R, \Phi\rangle = 0$ and hence $R = 0$. It follows that the restriction of [ ] to $\mathbb{R}\Gamma_s$ is injective.

Let $R \in \mathbb{R}\Gamma_s$ and $\Phi$ smooth. Using the identity $(f^n)^*[R] = d_s^n[R]$, we get

$$\begin{aligned} \langle R, \Phi\rangle &= d_s^{-n}\langle R, (f^n)_*\Phi\rangle \\ &= \int [R] \wedge (c_0 + d_s^{-1}c_1 + \cdots + d_s^{-n}c_n) + d_s^{-n}\langle R, \Phi_n\rangle. \end{aligned}$$

Since $R$ is PB, when $n \to \infty$, we get $\langle R, \Phi\rangle = \int [R] \wedge c_\Phi$ with $c_\Phi := \sum_{n\geq 0} d_s^{-n}c_n$. Following Proposition 2.1, $c_\Phi$ depends continuously on $\Phi$. Hence, we can extend $R$ to a continuous linear form on $\Phi \in \mathrm{DSH}^{k-s}(X)$ by

$$\langle R, \Phi\rangle := c_\Phi.$$

Hence $R$ is PC.

We show that the restriction of [ ] to $\mathbb{R}\mathcal{C}(T)$ is injective. Let $R \in \mathbb{R}\mathcal{C}(T)$ be a current such that $[R] = 0$. We have to prove that $R = 0$. We can write $R = R^+ - R^-$ with $R^\pm$ positive closed currents such that $R^\pm \leq cT$ for a constant $c > 0$. Define $R_n^\pm := d_s^n(f^n)_* R^\pm$ and $R_n := R_n^+ - R_n^-$. We have $R_n^\pm \leq c d_s^n (f^n)_* T = cT$. For a smooth test form $\Phi$, we have $|\langle T, \Phi_n^\pm\rangle| \lesssim (d_s - \epsilon)^n$. The domination of $R_n^\pm$ and the negativity of $\Phi_n^\pm$ imply that $|\langle R_n, \Phi_n^\pm\rangle| \lesssim (d_s - \epsilon)^n$. Since $[R_n] = 0$, we obtain from (3) that

$$|\langle R, \Phi\rangle| = d_s^{-n}|\langle (f^n)^* R_n, \Phi\rangle| = d_s^{-n}|\langle R_n, \Phi_n\rangle| \lesssim d_s^{-n}(d_s - \epsilon)^n.$$

Hence $\langle R, \Phi\rangle = 0$ and $R = 0$. This completes the proof of Theorem 4.1. □

**Corollary 4.2** *Let $f$ be a holomorphic automorphism of a compact Kähler manifold $X$ of dimension $k$. Assume that the dynamical degrees of $f$ are all distinct. Then for every $s$, $1 \leq s \leq k$, there exists a non-zero PC positive closed $(s, s)$-current $T_s$ such that $f^* T_s = c_s T_s$ with $c_s > 0$.*

**Proof.** The hypothesis implies, thanks to the Khovanskii-Tessier-Gromov convexity theorem [31, 36, 27] (see Section 2.3), the existence of $m$ such that

$$1 < d_1 < d_2 < \cdots < d_m > d_{m+1} > \cdots > d_k = 1.$$

Using Theorem 4.1, we construct the current $T_s$ such that $f^*T_s = d_sT_s$ for $1 \leq s \leq m$. The current $T_1$ can be constructed as in Corollary 3.5. We now construct the other currents by induction using Theorem 3.1 for $f^{-1}$. We construct $(1,1)$-currents $S_i$, $1 \leq i \leq k-m$, with Hölder continuous potentials and invariant currents $T_s$, $m+1 \leq s \leq k$, of the form $T_s = T_m \wedge S_1 \wedge \ldots \wedge S_{s-m}$. These currents satisfy $f^*T_s = c_sT_s$, $c_s > 0$. Since $f$ is an automorphism, we necesarily have $c_k = 1$.

In order to apply inductively Theorem 3.1 for $f^{-1}$, we need only to verify that the first dynamical degree $\lambda_1(T_s)$ of $f^{-1}$, relative to $T_s$, is strictly larger than 1 for $m \leq s \leq k-1$. Following the last inequality of Proposition 2.6, it is sufficient to prove that $c_s > 1$ for $m \leq s \leq k-1$. We have for every $\epsilon > 0$

$$\begin{aligned} c_s^{-n} &\lesssim \int (f^n)_* T_s \wedge \omega^{k-s} = d_m^{-n} \int T_m \wedge (f^n)_*(S_1 \wedge \ldots \wedge S_{s-m}) \wedge \omega^{k-s} \\ &= d_m^{-n} \int [S_1] \wedge \ldots \wedge [S_{s-m}] \wedge (f^n)^*[T_m \wedge \omega^{k-s}] \lesssim d_m^{-n}(d_{k-s+m} + \epsilon)^n. \end{aligned}$$

It follows that $c_s > 1$ for $m \leq s \leq k-1$. This completes the induction step.

One can check that the wedge product of a PC positive closed current with a current of bidegree $(1,1)$ with continuous potential is always PC. $\square$

# 5 Mixing of the equilibrium measure

In this section, using the methods developed above, we can construct, for automorphisms with distinct dynamical degrees, an equilibrium measure which is PC and mixing. We get the following result.

**Theorem 5.1** *Let $f$ be a holomorphic automorphism of a compact Kähler manifold $X$ of dimension $k$. Assume that the dynamical degrees of $f$ are all distinct. Then $f$ admits a mixing PC invariant measure $\mu$. Moreover, $\mu$ gives no mass to sets with small Hausdorff dimension.*

We need the following variation of Ahlfors's estimate (see [2, 35]).

**Lemma 5.2** *Let $f$ be a holomorphic automorphism of $X$. Let $T$ be a positive closed $(s,s)$-current such that $f^*T = \lambda_T T$ with $\lambda_T > 0$. Assume that $\lambda_1(T) < 1$. Then for every smooth function $\psi \geq 0$, the limit values of the sequence $S_n := \lambda_T^{-n}(f^n)^*(\psi T)$ are positive closed currents. Moreover, $\|\mathrm{d}S_n\| \to 0$ and $\|\mathrm{dd^c}S_n\| \to 0$. If $S_{n_i} \rightharpoonup S$ and if $\sigma$ is a closed $(1,1)$-current with a continuous potential, then $S_{n_i} \wedge \sigma \rightharpoonup S \wedge \sigma$.*

**Proof.** Let $\theta$ be a continuous $(0,1)$-form. The Cauchy-Schwarz inequality implies that

$$\begin{aligned} A_n &:= \Big|\int (f^n)^*(\partial\psi)\wedge T\wedge\theta\wedge\omega^{k-s-1}\Big| \\ &\leq \Big|\int (f^n)^*(\partial\psi\wedge\overline{\partial\psi})\wedge T\wedge\omega^{k-s-1}\Big|^{1/2}\Big|\int \theta\wedge\overline{\theta}\wedge T\wedge\omega^{k-s-1}\Big|^{1/2} \\ &\leq c\Big|\int (f^n)^*\omega\wedge T\wedge\omega^{k-s-1}\Big|^{1/2}\Big|\int \omega\wedge T\wedge\omega^{k-s-1}\Big|^{1/2} \end{aligned}$$

if $i\partial\psi\wedge\overline{\partial\psi}$ and $i\theta\wedge\overline{\theta}$ are bounded by $c\omega$, $c>0$. It follows that $A_n \lesssim (\lambda_1(T)+\epsilon)^{n/2}$. Since $\lambda_1(T)<1$, we have $\lim A_n = 0$. As a consequence, $\lim \|\partial S_n\| = 0$, hence $\lim \|\mathrm{d}S_n\| = 0$.

To estimate $\|\mathrm{dd^c}S_n\|$, one has just to observe that for $c>0$ large enough

$$-c(f^n)^*\omega\wedge T \leq \mathrm{dd^c}(f^n)^*\psi\wedge T \leq c(f^n)^*\omega\wedge T.$$

Let $u$ be a local continuous potential of $\sigma$ and $\theta$ be a test form. Define $\psi_n := (f^n)^*\psi$. For the last assertion of this lemma, we have

$$\begin{aligned} \langle \psi_n T\wedge \mathrm{dd^c}u, \theta\rangle &= \langle \mathrm{dd^c}(uT), \psi_n\theta\rangle \\ &= \langle \mathrm{dd^c}(\psi_n T), u\theta\rangle + \langle \mathrm{d}(\psi_n T), u\mathrm{d^c}\theta\rangle - \\ &\quad -\langle \mathrm{d^c}(\psi_n T), u\mathrm{d}\theta\rangle + \langle \mathrm{dd^c}(u\psi_n T), \theta\rangle \end{aligned}$$

The first three terms tend to zero. Hence, $S_{n_i}\wedge\sigma \rightharpoonup S\wedge\sigma$. □

**Proof of Theorem 5.1.** We construct invariant currents $T_s$ as in Corollary 4.2. We choose an extremal current $T_m$ in $\Gamma_m$. We can write $T_s = T_m\wedge S_1\wedge \ldots\wedge S_{s-m}$ for $s\geq m+1$, where $S_i$ are closed $(1,1)$-currents with Hölder continuous potentials. Define $\mu := T_k$. Hence $\mu$ is PC. The estimate of Hausdorff dimension of $\mu$ uses classical arguments [14, 35]. If the potentials of $S_j$, $1\leq j\leq k-m$, are $\alpha_j$-Hölder continuous, then $\mu$ gives no mass to sets whose Hausdorff dimension is smaller than $\alpha_1+\cdots+\alpha_{k-m}$.

We show first that $\mu$ is ergodic. Let $\psi\geq 0$ be a smooth test function. Let $\tau$ be the limit of a sequence of currents $n_i^{-1}\sum_{j=1}^{n_i}(\psi\circ f^j)T_m$. It is clear that $f^*\tau = d_m\tau$ and $\tau\leq \|\psi\|_\infty T_m$. Lemma 5.2 implies that $\tau$ is closed. Hence $\tau\in\Gamma_m$. Since $T_m$ is extremal in $\Gamma_m$, we have $\tau = cT_m$ for a constant $c$. We can now apply inductively Lemma 5.2. Since the currents $S_r$ have continuous potentials and $\lambda_1(T_s)<1$ for $m\leq s\leq k-1$, $n_i^{-1}\sum_{j=1}^{n_i}(\psi\circ f^j)\mu$ converge to $c\mu$. The invariance property of $\mu$ implies

that $c = \|\mu\|^{-1}\int \psi \mathrm{d}\mu$. This constant does not depend on the sequence $(n_i)$. Consequently, $n^{-1}\sum_{j=1}^n (\psi\circ f^j)\mu$ converge to $c\mu$. Hence $\mu$ is ergodic.

We now prove that $\mu$ is mixing, which means $(f^n)^*\psi\mu \to c\mu$, $c = \|\mu\|^{-1}\int \psi \mathrm{d}\mu$, for every smooth function $\psi$. Let $M$ denote the set of measures which are limite values of the sequence $(f^{n*}\psi)\mu$ for some smooth function $\psi$. Since $\mathcal{C}(T_m)$ is finite dimensional, Lemma 5.2 implies that $M$ is a finite dimensional space which contains $\mu$ and which is invariant under $f^*$ and $f_*$. Let $E$ denote the space of functions $\varphi \in \mathrm{L}^2(\mu)$ such that $\int \varphi \mathrm{d}\mu' = 0$ for every $\mu' \in M$ and $E^\perp$ its orthogonal. Observe that these spaces are invariant under $f^*$, $f_*$ and that we have $\dim E^\perp = \dim M$. Moreover, in $E$, every function can be approximated by smooth ones.

We show that $\dim E^\perp = 1$. Since $f^*$ and $f_*$ preserve the scalar product in $\mathrm{L}^2(\mu)$, every eigenvalue of $f^*$ or $f_*$ has modulus equal to 1. Let $\varphi$ be an eigenvector of $f_*$ associated to an eigenvalue $\lambda$. We have $f_*|\varphi| = |\varphi|$. The ergodicity of $\mu$ implies that $|\varphi|$ is constant. In particular, $\varphi^n \in \mathrm{L}^2(\mu)$ for every $n \geq 1$ and we have $f_*\varphi^n = \lambda^n\varphi^n$. We claim that $E$ does not contain any eigenvector. Otherwise, there is a function $\varphi \in E \setminus \{0\}$ such that $f_*\varphi = \lambda\varphi$ with a $\lambda$ such that $|\lambda| = 1$. We have for every smooth function $\psi$:

$$|\langle \varphi\mu, \psi\rangle| = |\langle (f^n)_*\varphi\mu, \psi\rangle| = |\langle (f^{n*}\psi)\mu, \varphi\rangle| \to 0.$$

The last relation follows from the definition of $E$. We need of course to approach $\varphi$ by smooth functions in $E$. We get that $\varphi\mu = 0$, hence $\varphi = 0$. A contradiction.

Let $\varphi$ be an eigenvector of $f_*$ in $E^\perp$ associated to an eigenvalue $\lambda$. Then, $\varphi^n$ belongs to $E^\perp$ and is an eigenvector associated to $\lambda^n$ for every $n \geq 1$. Since $\dim E^\perp$ is finite, $\lambda$ is a root of unity. We have $f^{n*}\varphi = \varphi$ for some $n \geq 1$. Since $\mu$ is ergodic, $\varphi$ is constant. Hence, $\lambda = 1$. Since $f^*_{|E^\perp}$ preserves the scalar product, $\dim E^\perp = 1$ and $M$ is generated only by $\mu$. If $(f^{n_i*}\psi)\mu \to c\mu$, we have $c = \|\mu\|^{-1}\int \psi \mathrm{d}\mu$. This constant does not depend on $(n_i)$. Hence $(f^{n*}\psi)\mu \to c\mu$ and $\mu$ is mixing. □

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